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Ratio Asymptotics for Orthogonal Matrix Polynomials with Unbounded Recurrence Coefficients[1]

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We study ratio asymptotic behaviour for orthogonal matrix polynomials with unbounded recurrence coefficients. © 2001 Academic Press

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1. INTRODUCTION

Ratio asymptotic results give the asymptotic behaviour of the ratio of two consecutive polynomials, p_n and p_{n+1}, orthogonal with respect to a positive measure μ. Since the first systematic study was accomplished in 1979 by P. Nevai (see [N, Theorem 13, p. 33]), a lot of work has been devoted to obtaining asymptotic properties of this type from the recurrence coefficients $(a_n)_n$ and $(b_n)_n$. Nevai studied precisely the case when these recurrence coefficients have finite limits. Asymptotically periodic recurrence coefficients with finite accumulation points have been studied by Van Assche and Geronimo (see [V1, Chap. 2; V3; GV]). Van Assche also studied the case of unbounded recurrence coefficients (see [V1, Theorem 4.10; V2]), and recently a technique to find the ratio asymptotics of a polynomial s_n and the nth polynomial p_n, orthonormal with respect to a positive measure, has been developed by one of the authors (see [D1]).

During the past few years, some important results in the theory of orthogonal polynomials have been extended to orthogonal matrix polynomials with the consequence that this topic of matrix orthogonality is receiving an increasing amount of interest (see [BB, D2–DV, JCN, JDP, JD, SV]). In particular in [D6], one of the authors has extended ratio asymptotics to orthogonal matrix polynomials with convergent matrix recurrence coefficients.

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The purpose of this paper is to study the case when the matrix recurrence coefficients are unbounded.

We consider an $N \times N$ positive definite matrix of measures W (for any Borel set $A \subset \mathbb{R}$, $W(A)$ is a positive semidefinite numerical matrix), having moments of every order, i.e., the matrix integral $\int_{\mathbb{R}} t^n \, dW(t)$ exists for every nonnegative integer n.

Assuming that $\int P(t) \, dW(t) \, P^*(t)$ is nonsingular for every matrix polynomial P with nonsingular leading coefficient, the matrix inner product defined in the usual way by W in the space of matrix polynomials has a sequence of orthonormal matrix polynomials $(P_n)_n$, satisfying

$$\int P_n(t) \, dW(t) \, P_m^*(t) = \delta_{n,m} I, \qquad n, m \geqslant 0.$$

$P_n(t)$ is a matrix polynomial of degree n, with a non-singular leading coefficient and is defined up to a multiplication on the left by a unitary matrix.

As in the scalar case, the sequence of orthonormal matrix polynomials $(P_n)_n$ satisfies a three-term recurrence relation

$$tP_n(t) = A_{n+1} P_{n+1}(t) + B_n P_n(t) + A_n^* P_{n-1}(t), \qquad n \geqslant 0, \tag{1.1}$$

where $P_{-1}(t) = \theta$, $P_0(t) \in \mathbb{C}^{N \times N} \setminus \{\theta\}$, A_n are non-singular matrices and B_n are hermitian. Here and in the rest of this paper, we write θ for the null matrix, the dimension of which can be determined from the context. We remark that the polynomials $R_n(t) = U_n P_n(t)$, with $U_n U_n^* = I$ are also orthonormal with respect to the same positive definite matrix of measures with respect to which the $(P_n)_n$ are orthonormal, and satisfy a three-term recurrence relation as (1.1) with coefficients $U_{n-1} A_n U_n^*$ instead of A_n and $U_n B_n U_n^*$ instead of B_n. This three-term recurrence relation characterizes the orthonormality of a sequence of matrix polynomials with respect to a positive definite matrix of measures (see, for instance, [AN, DL1]). In [D2, D3, DV] a very close relationship between orthogonal matrix polynomials and scalar polynomials satisfying a higher order recurrence relation has been established. This relationship has been used to show that matrix orthogonality is going to be a useful tool to solve certain problems of scalar orthogonality (see [D2, Sect. 5]).

In this paper we assume that the matrix recurrence coefficients are diverging in a particular way: we will suppose that there exists a sequence $(C_n)_n$ of positive definite matrices such that

$$\lim_n C_n^{-1/2} A_n C_n^{-1/2} = A, \qquad \lim_n C_n^{-1/2} B_n C_n^{-1/2} = B,$$
$$\lim_n C_n^{-1/2} C_{n-1}^{1/2} = I. \tag{1.2}$$

When unbounded coefficients are considered in the scalar case (assuming the same hypothesis given by (1.2)), the ratio asymptotic behaviour is then obtained for the scaled polynomials $p_n(c_n z)$. So, in the matrix case, the first question to be solved is how the scaled matrix polynomial $P(Cx)$, with P a matrix polynomial and C a matrix, should be defined. Given a matrix polynomial

$$P(z) = \sum_{k=1}^{n} A_k z^k,$$

we have found in the literature two equally natural definitions of $P(Cx)$ (see [HJ2, p. 384]),

$$P_l(Cx) = \sum_{k=0}^{n} A_k C^k x^k, \tag{1.3}$$

and

$$P_r(Cx) = \sum_{k=0}^{n} C^k A_k x^k. \tag{1.4}$$

However, we now show that there is a large range of possibilities including these two ones. To avoid any confusion, we will use the notation $P(C; x)$ instead of $P(Cx)$ for the scaled matrix polynomial.

Indeed, let us consider matrix polynomials of a matrix variable, that is, finite combinations of the form

$$\mathbf{P}(T) = \sum_{\textit{finite}} \alpha_{n_1} T^{n_1} \alpha_{n_2} \cdots \alpha_{n_k} T^{n_k} \alpha_{n_k+1},$$

where T is the matrix variable given by

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1N} \\ t_{21} & t_{22} & \cdots & t_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ t_{N1} & t_{N2} & \cdots & t_{NN} \end{pmatrix}$$

and the α's are matrices of size $N \times N$. For a given matrix C of size $N \times N$, we define $\mathbf{P}(C)$ in the natural way

$$\mathbf{P}(C) = \sum_{\textit{finite}} \alpha_{n_1} C^{n_1} \alpha_{n_2} \cdots \alpha_{n_k} C^{n_k} \alpha_{n_k+1}.$$

The matrix polynomial $P(z)$ can be obtained from many different matrix polynomials $\mathbf{P}$ of a matrix variable just by replacing T by zI in $\mathbf{P}(T)$. For instance, let us consider the matrix polynomial of a matrix variable

$$\mathbf{P}_1(T) = \sum_{k=0}^{n} A_k T^k,$$

then we easily have that $P(z) = \mathbf{P}_1(zI)$. But also $P(z) = \mathbf{P}_2(zI)$, where

$$\mathbf{P}_2(T) = \sum_{k=0}^{n} T^k A_k,$$

and it is clear that, due to the noncommutativity of the matrix product, in general $\mathbf{P}_1 \neq \mathbf{P}_2$ as matrix polynomials of a matrix variable. Of course, there are many other possibilities than $\mathbf{P}_1$ and $\mathbf{P}_2$ to obtain P; for instance if we consider B, a square root of A_1, so that $A_1 = BB$, then, we have that $P(z) = \mathbf{P}_3(zI)$ where

$$\mathbf{P}_3(T) = A_0 + BTB + \sum_{k=2}^{n} T^k A_k,$$

and clearly, in general, $\mathbf{P}_3$ is different from $\mathbf{P}_1$ or $\mathbf{P}_2$ as matrix polynomials of a matrix variable.

We can define in an equally natural way the scaled matrix polynomial $P(C; z)$, using each matrix polynomial of one matrix variable $\mathbf{P}$, from which P is obtained in the way explained before, just by putting $P(C; z) = \mathbf{P}(Cz)$. $\mathbf{P}_1$ and $\mathbf{P}_2$ give the definitions (1.3) and (1.4), respectively, which are usually considered in the literature. What definition of $P(C; z)$ should we consider to study the asymptotic behaviour for orthogonal matrix polynomials with unbounded recurrence coefficients? The key is given by the three-term matrix recurrence relation (1.1). Indeed, using the recurrence coefficients $(A_n)_n$, $(B_n)_n$, we can define a sequence of matrix polynomials of one matrix variable as

$$\mathrm{T}\mathbf{P}_n(T) = A_{n+1}\mathbf{P}_{n+1}(T) + B_n\mathbf{P}_n(T) + A_n^*\mathbf{P}_{n-1}(T), \qquad n \geqslant 0, \quad (1.5)$$

with initial conditions $\mathbf{P}_{-1}(T) = \theta$, $\mathbf{P}_0(T) = P_0$. It is clear that $\mathbf{P}_n(zI) = P_n(z)$, $n \geqslant 0$. We then define $P_n(C; z) = \mathbf{P}_n(Cz)$. This is the choice we find the most natural, in the context of orthogonal matrix polynomials, to define the scaled polynomial $P_n(C_n; z)$.

As a consequence of our definition, the scaled matrix polynomial $P_n(C_n; z)$ satisfies some important properties related to orthogonality. In fact, although the sequence of scaled polynomials $(P_n(C_n; z))_n$ is not orthogonal with respect to a positive definite matrix of measures (except for trivial examples), each scaled polynomial $P_n(C_n; z)$ is the nth orthogonal polynomial with respect to a certain varying matrix weight W_n (see Section 3). In fact this allows us to compute $P_n(C_n; z)$ easily without using the matrix polynomials $\mathbf{P}_n$ of one matrix variable. This implies, in particular, that the zeros of $P_n(C_n; z)$ are real with multiplicity not greater than N. This may not be the case if any other definition of a scaled matrix polynomial is used, as it can easily be checked for the choices (1.3) and (1.4), where complex zeros, or zeros with multiplicities larger than any given number, can appear.

To establish our main result we need to introduce the matrix analogs of the Chebyshev polynomials of the second kind (see [D6]): We associate to two given matrices A (nonsingular), and B (hermitian), the orthonormal matrix polynomials $(U_n^{A,B})_n$ defined by the recurrence formula

$$tU_n^{A,B}(t) = A^* U_{n+1}^{A,B}(t) + BU_n^{A,B}(t) + AU_{n-1}^{A,B}(t), \qquad n \geqslant 0, \tag{1.6}$$

with initial conditions $U_0^{A,B}(t) = I$, $U_{-1}^{A,B}(t) = \theta$.

We are now ready to establish the main result of this paper:

THEOREM 1.1. *Let $(P_n)_n$ be orthonormal matrix polynomials satisfying the three-term recurrence relation* (1.1). *Suppose that there exists a sequence of positive definite matrices $(C_n)_n$ such that* (1.2) *holds with A nonsingular and B hermitian, and consider the scaled matrix polynomials $P_n(C_n; z)$ defined by using* (1.5). *We write Δ_n for the set of zeros of $P_n(C_n; z)$ and $\Gamma = \bigcap_{N \geqslant 0} M_N$, where $M_N = \bigcup_{n \geqslant N} \Delta_n$. Then:*

(a) *$\Delta_n \subset \mathbb{R}$, and if we assume that the matrix sequence $(C_n)_n$ is increasing then Γ is a compact set.*

(b)
$$\lim_{n \to \infty} C_n^{1/2} P_{n-1}(C_n; z)\, P_n^{-1}(C_n; z)\, A_n^{-1} C_n^{1/2}$$
$$= \int \frac{dW_{A,B}(t)}{z-t}, \qquad z \in \mathbb{C} \setminus \Gamma, \tag{1.7}$$

where $W_{A,B}$ is the matrix weight for the Chebyshev matrix polynomials of the second kind defined by (1.6). *Moreover, the convergence is uniform for z on compact subsets of $\mathbb{C} \setminus \Gamma$.*

For each non-negative integer k, the sequence $(P_n(C_k; z))_n$ is orthogonal with respect to a certain varying matrix weight W_k, and actually, Theorem 1.1 is a consequence of a more general theorem (see Theorem 2.1) on ratio asymptotics for orthogonal matrix polynomials with varying recurrence coefficients (see [KV], for its scalar version).

We complete the paper studying the case when A is singular (Section 4) and giving some examples (Section 5).

2. RATIO ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS WITH VARYING RECURRENCE COEFFICIENTS

We consider orthonormal matrix polynomials $(R_{n,k})_n$, $k=1, 2, ...$, depending on a parameter k, and given by the recurrence relation

$$tR_{n,k}(t)=A_{n+1,k}R_{n+1,k}(t)+B_{n,k}R_{n,k}(t)+A^*_{n,k}R_{n-1,k}(t), \qquad n\geqslant 0, \quad (2.1)$$

with initial conditions $R_{0,k}(t)$ a nonsingular matrix and $R_{-1,k}(t)=\theta$. Without loss of generality, we assume $R_{0,k}=I$.

The sequence of matrix polynomials $(R_{n,k})_n$ is orthonormal with respect to a certain measure which we denote by W_k.

Our main result in this section is the following.

THEOREM 2.1. *Let $(R_{n,k})_n$ be a sequence of orthonormal matrix polynomials depending on a parameter k, $k=1, 2, ...,$ satisfying the three-term recurrence relation* (2.1). *Let $(n_m)_m$, $(k_m)_m$ be two increasing sequences of positive integers and assume that there exist two matrices A nonsingular and B hermitian such that for all $l\geqslant 0$*

$$\lim_{m\to\infty} A_{n_m-l,k_m}=A, \qquad \lim_{m\to\infty} B_{n_m-l,k_m}=B. \quad (2.2)$$

We write Δ_m for the set of zeros of R_{n_m,k_m} and $\Gamma=\bigcap_{N\geqslant 0} M_N$, where $M_N=\bigcup_{m\geqslant N}\Delta_m$. Then

$$\lim_{m\to\infty} R_{n_m-1,k_m}(z)\,R^{-1}_{n_m,k_m}(z)\,A^{-1}_{n_m,k_m}=\int\frac{dW_{A,B}(t)}{z-t}, \qquad \textit{for} \quad z\in\mathbb{C}\backslash\Gamma, \quad (2.3)$$

where $W_{A,B}$ is the matrix weight for the Chebyshev matrix polynomials of the second kind defined by (1.6). *Moreover, the convergence is uniform for z on compact subsets of $\mathbb{C}\backslash\Gamma$.*

Proof. We proceed as in the proof of Theorem 1.1 in [D6].

First of all, we prove that

$$\lim_{m\to\infty} R_{n_m-1,k_m}(z)\,R^{-1}_{n_m,k_m}(z)\,A^{-1}_{n_m,k_m}=\int\frac{dW_{A,B}(t)}{z-t}$$

for $z\in\mathbb{C}\backslash\Gamma$.

To do that, we consider the following matrices of discrete measures,

$$\mu_{n,k}=\sum_{j=1}^{m}\delta_{x_{n,k,j}}R_{n-1,k}(x_{n,k,j})\,\Gamma_{n,k,j}R^*_{n-1,k}(x_{n,k,j}),$$

where $x_{n,k,j}$, $j=1, ..., m$, are the different zeros of the polynomial $R_{n,k}$ and the matrix $\Gamma_{n,k,j}$ is given by

$$\Gamma_{n,k,j} = \frac{l_k}{(\det(R_{n,k}(t)))^{(l_j)}(x_{n,k,j})} (\operatorname{Adj}(R_{n,k}(t)))^{(l_j-1)}(x_{n,k,j})\, Q_{n,k}(x_{n,k,j}),$$

where l_j is the multiplicity of $x_{n,k,j}$ and $(Q_{n,k})_n$ the sequence of polynomials of the second kind associated to $(R_{n,k})_n$. $\Gamma_{n,k,j}$ are the matrix weights in the quadrature formula for the polynomials $(R_{n,k})_n$ associated to $x_{n,k,j}$ and $l_j \leqslant N$ (see [D4]).

From the quadrature formula we can prove that $\int d\mu_{n,k}(t) = I$,

$$\begin{aligned}\int d\mu_{n,k}(t) &= \sum_{j=1}^{m} R_{n-1,k}(x_{n,k,j})\, \Gamma_{n,k,j} R^*_{n-1,k}(x_{n,k,j})\\ &= \int R_{n-1,k}(t)\, dW_k(t)\, R^*_{n-1,k}(t) \qquad (2.4)\\ &= I.\end{aligned}$$

In order to obtain (2.3) we proceed in several steps:

Step 1. For two given nonnegative integers n and k, we have

$$R_{n-1,k}(z)\, R^{-1}_{n,k}(z)\, A^{-1}_{n,k} = \int \frac{d\mu_{n,k}(t)}{z-t}.$$

This follows analogously as in the *first step* in the proof of Theorem 1 in [D6].

Note that, according to Step 1, we have to prove

$$\lim_{m\to\infty} \int \frac{d\mu_{n_m,k_m}(t)}{z-t} = \int \frac{dW_{A,B}(t)}{z-t}, \qquad \text{for} \quad z \in \mathbb{C}\setminus\Gamma.$$

Step 2. Let us consider the Chebyshev matrix polynomials of the second kind $(U_n^{A,B})_n$ defined by (1.6). Then

$$\lim_{m\to\infty} \int U_l^{A,B}(t)\, d\mu_{n_m,k_m}(t) = \begin{cases} I & \text{for} \quad l=0,\\ \theta & \text{for} \quad l\neq 0.\end{cases}$$

Indeed, we can write

$$U_l^{A,B}(t)\, R_{n-1,k}(t) = S_{l,n-1,k}(t)\, R_{n,k}(t) + \sum_{i=1}^{n} \Delta_{i,l,n-1,k} R_{n-i,k}(t). \qquad (2.5)$$

According to the definition of the discrete measures $\mu_{n,k}$, using the expression (2.5), and proceeding as in the proof of Theorem 1 in [D6] we obtain that

$$\int U_l^{A,B}(t)\,d\mu_{n,k}(t)=\Delta_{1,l,n-1,k}.$$

So, Step 2 will follow if we prove that

$$\lim_{m\to\infty}\Delta_{j,l,n_m-1,k_m}=\begin{cases} I & \text{for } j=l+1,\\ \theta & \text{for } j\neq l+1.\end{cases}$$

We use induction on l. When $l=0$ the result is immediate. Now suppose the result is valid up to l. The three-term recurrence relation for the matrix polynomials $(U_n^{A,B})_n$ gives that

$$U_{l+1}^{A,B}(t)\,R_{n-1,k}(t)=A^{*-1}(tU_l^{A,B}(t)-BU_l^{A,B}(t)-AU_{l-1}^{A,B}(t))\,R_{n-1,k}(t).$$

The expression (2.5) and the three-term recurrence relation for $(U_n^{A,B})$ gives that

$$\begin{aligned}\Delta_{j,l+1,n-1,k}=A^{*-1}(&\Delta_{j,l,n-1,k}B_{n-j,k}+\Delta_{j-1,l,n-1,k}A^*_{n-j+1,k}\\ &+\Delta_{j+1,l,n-1,k}A_{n-j,k})-A^{*-1}B\,\Delta_{j,l,n-1,k}\\ &-A^{*-1}A\,\Delta_{j,l-1,n-1,k}.\end{aligned}\tag{2.6}$$

For $j\geqslant l+3$ or $j\leqslant l-1$ the induction hypothesis shows that

$$\lim_{m\to\infty}\Delta_{j,l+1,n_m-1,k_m}=\theta.$$

We study the cases $j=l$, $j=l+1$, and $j=l+2$ separately:

Case 1. $j=l$.

$$\begin{aligned}\lim_{m\to\infty}\Delta_{l,l+1,n_m-1,k_m}&=\lim_{m\to\infty}A^{*-1}(\Delta_{l,l,n_m-1,k_m}B_{n_m-l,k_m}\\ &\quad+\Delta_{l-1,l,n_m-1,k_m}A^*_{n_m-l+1,k_m}\\ &\quad+\Delta_{l+1,l,n_m-1,k_m}A_{n_m-l,k_m})\\ &\quad-\lim_{m\to\infty}A^{*-1}B\,\Delta_{l,l,n_m-1,k_m}\\ &\quad-\lim_{m\to\infty}A^{*-1}A\,\Delta_{l,l-1,n_m-1,k_m}\\ &=A^{*-1}(A-A)=\theta,\end{aligned}$$

since, from (2.2), we have that $\lim_{m\to\infty}A_{n_m-l,k_m}=A$.

Case 2. $j=l+1$.

$$\begin{aligned}\lim_{m\to\infty} \Delta_{l+1,\,l+1,\,n_m-1,\,k_m} &= \lim_{m\to\infty} A^{*-1}(\Delta_{l+1,\,l,\,n_m-1,\,k_m} B_{n_m-l-1,\,k_m}\\ &\quad+\Delta_{l,\,l,\,n_m-1,\,k_m} A^*_{n_m-l,\,k_m}\\ &\quad+\Delta_{l+2,\,l,\,n_m-1,\,k_m} A_{n_m-l-1,\,k_m})\\ &\quad-\lim_{m\to\infty} A^{*-1} B\,\Delta_{l+1,\,l,\,n_m-1,\,k_m}\\ &\quad-\lim_{m\to\infty} A^{*-1} A\,\Delta_{l+1,\,l-1,\,n_m-1,\,k_m}\\ &= A^{*-1}(B-B)=\theta,\end{aligned}$$

since, from (2.2), we have that $\lim_{m\to\infty} B_{n_m-l-1,\,k_m}=B$.

Case 3. $j=l+2$.

$$\begin{aligned}\lim_{m\to\infty} \Delta_{l+2,\,l+1,\,n_m-1,\,k_m} &= \lim_{m\to\infty} A^{*-1}(\Delta_{l+2,\,l,\,n_m-1,\,k_m} B_{n_m-l-2,\,k_m}\\ &\quad+\Delta_{l+1,\,l,\,n_m-1,\,k_m} A^*_{n_m-l-1,\,k_m}\\ &\quad+\Delta_{l+3,\,l,\,n_m-1,\,k_m} A_{n_m-l-2,\,k_m})\\ &\quad-\lim_{m\to\infty} A^{*-1} B\,\Delta_{l+2,\,l,\,n_m-1,\,k_m}\\ &\quad-\lim_{m\to\infty} A^{*-1} A\,\Delta_{l+2,\,l-1,\,n_m-1,\,k_m}\\ &= A^{*-1}A^*=I,\end{aligned}$$

since, from (2.2), we obtain $\lim_{m\to\infty} A_{n_m-l-1,\,k_m}=A$. Thus, Step 2 has been proved.

We are now ready to prove that

$$\lim_{m\to\infty}\int\frac{d\mu_{n_m,\,k_m}(t)}{z-t}=\int\frac{dW_{A,\,B}(t)}{z-t},\qquad \text{for}\quad z\in\mathbb{C}\backslash\Gamma.$$

We will use the so-called *method of moments*: suppose μ_n and μ are probability measures on $\mathbb{R}$ with moments of every order and that μ has compact support. If $(r_n)_n$ is a sequence of polynomials, r_n of degree n, and $\lim_n \int r_k(t)\,d\mu_n(t)=\int r_k(t)\,d\mu(t)$ for $k=0, 1, 2, ...$, then μ_n converges weakly to μ (see [F]). This method can easily be extended to positive definite matrices of measures. *Weak convergence* has the usual meaning: a sequence of matrices of measures $(\mu_n)_n$ on a metric space X converges weakly to μ if $\lim_n \int f\,d\mu_n=\int f\,d\mu$ for every continuous and bounded function $f\colon X\to\mathbb{C}^{N\times N}$.

The sequence of matrix polynomials $(U_l^{A,B})_l$ is orthonormal with respect to $W_{A,B}$ which has compact support (see [D6, Lemma 2.1]), and hence

$$\int U_l^{A,B}(t)\, dW_{A,B}(t) = \begin{cases} I & \text{for } \quad l=0, \\ \theta & \text{for } \quad l \neq 0. \end{cases}$$

Using Step 2 and by applying the *method of moments* we get that the sequence $(\mu_{n_m, k_m})_m$ converges weakly to $W_{A,B}$ and so, since $1/(z-t)$ is bounded for $z \in \mathbb{C} \backslash \Gamma$, we get that

$$\lim_{m \to \infty} \int \frac{d\mu_{n_m, k_m}(t)}{z-t} = \int \frac{dW_{A,B}(t)}{z-t}.$$

The uniform convergence on compact sets of $\mathbb{C} \backslash \Gamma$ follows from the Stieltjes–Vitali Theorem. ∎

3. PROOF OF THEOREM 1.1

Let $(C_n)_n$ be a sequence of $N \times N$ positive definite matrices and let $(P_n)_n$ be a sequence of orthonormal polynomials satisfying the three-term recurrence relation (1.1). In order to apply Theorem 2.1, we consider the scaled sequence of matrix polynomials $(P_n(C_k; t))_n$. Taking into account (1.5), we have

$$C_k t P_n(C_k; t) = A_{n+1} P_{n+1}(C_k; t) + B_n P_n(C_k; t) + A_n^* P_{n-1}(C_k; t), \qquad n \geqslant 0,$$

and so,

$$\begin{aligned} tC_k^{1/2} P_n(C_k; t) = {} & C_k^{-1/2} A_{n+1} C_k^{-1/2} C_k^{1/2} P_{n+1}(C_k; t) \\ & + C_k^{-1/2} B_n C_k^{-1/2} C_k^{1/2} P_n(C_k; t) \\ & + C_k^{-1/2} A_n^* C_k^{-1/2} C_k^{1/2} P_{n-1}(C_k; t), \qquad n \geqslant 0. \end{aligned} \tag{3.1}$$

For each k, write

$$\begin{aligned} R_{n,k}(t) &= C_k^{1/2} P_n(t; C_k); \\ A_{n,k} &= C_k^{-1/2} A_n C_k^{-1/2}; \\ B_{n,k} &= C_k^{-1/2} B_n C_k^{-1/2}. \end{aligned}$$

Hence, for each k, the sequence of matrix polynomials $(R_{n,k})_n$ satisfies the following three-term recurrence relation,

$$tR_{n,k}(t) = A_{n+1,k} R_{n+1,k}(t) + B_{n,k} R_{n,k}(t) + A_{n,k}^* R_{n-1,k}(t), \qquad n \geqslant 0,$$

with initial conditions $R_{0,k}(t) = C_k^{1/2}$ and $R_{-1,k}(t) = \theta$. The sequence of matrix polynomials $(R_{n,k})_n$ is then orthonormal with respect to a certain varying matrix of measures W_k.

Under the assumptions (1.2) it is easy to see that the limit condition (2.2) holds for $n_m = k_m = m$. Indeed, for $l \geqslant 0$

$$\lim_{n\to\infty} A_{n-l,n} = \lim_{n\to\infty} C_n^{-1/2} A_{n-l} C_n^{-1/2}$$

$$= \lim_{n\to\infty} C_n^{-1/2} C_{n-1}^{1/2} C_{n-1}^{-1/2} \cdots C_{n-l}^{1/2} C_{n-l}^{-1/2} A_{n-l}$$

$$C_{n-l}^{1/2} C_{n-l}^{-1/2} \cdots C_{n-1}^{-1/2} C_{n-1}^{1/2} C_n^{-1/2} = A.$$

In the same way we can prove that $\lim_{n\to\infty} B_{n-l,n} = B$, for $l \geqslant 0$.

We now show that the zeros of $P_n(C_n; t)$ are real. Indeed, $P_n(C_n; t) = C_n^{-1/2} R_{n,n}(t)$, and so the zeros of $P_n(C_n; t)$ are the zeros of $R_{n,n}(t)$. But these zeros are real because $R_{n,n}(t)$ is the nth orthonormal polynomial with respect to a certain matrix weight W_n.

Finally we prove that, assuming the matrix sequence $(C_n)_n$ to be increasing, the zeros of $P_n(C_n; t)$ are bounded (and so Γ is a compact set):

LEMMA 3.1. *Assume that the matrix sequence $(C_n)_n$ is increasing. Then there exists a positive constant $M > 0$, which does not depend on n, satisfying that if $x_{n,n,j}$ is a zero of $P_n(C_n; t)$ then $|x_{n,n,j}| < M$.*

Proof. We consider J, the N-Jacobi matrix associated to $(P_n)_n$, and $\mathcal{J}^{(k)}$, the N-Jacobi matrix associated to $(C_k^{1/2} P_n(C_k; \cdot))_n$ defined by

$$J = \begin{pmatrix} B_0 & A_1 & \theta & \theta & \cdots \\ A_1^* & B_1 & A_2 & \theta & \cdots \\ \theta & A_2^* & B_2 & A_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

and

$$\mathcal{J}^{(k)} = \begin{pmatrix} C_k^{-1/2} B_0 C_k^{-1/2} & C_k^{-1/2} A_1 C_k^{-1/2} & \theta & \theta & \cdots \\ C_k^{-1/2} A_1^* C_k^{-1/2} & C_k^{-1/2} B_1 C_k^{-1/2} & C_k^{-1/2} A_2 C_k^{-1/2} & \theta & \cdots \\ \theta & C_k^{-1/2} A_2^* C_k^{-1/2} & C_k^{-1/2} B_2 C_k^{-1/2} & C_k^{-1/2} A_3 C_k^{-1/2} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

In Lemma 2.1 of [DL1, p. 101] it is proved that the zeros of $P_n(t)$ are the eigenvalues of J_{nN} (truncated N-Jacobi matrix of dimension nN).

Analogously, the zeros of $P_n(C_n; t)$ are the eigenvalues of the matrix $\tilde{J}_{nN}^{(n)}$. Using the Gershgorin disk theorem for the location of eigenvalues, it is enough to show that the entries of the matrix $\tilde{J}_{nN}^{(n)}$ are bounded (independently of n). But the entries of $\tilde{J}_{nN}^{(n)}$ are of the form $C_n^{-1/2} A_m C_n^{-1/2}$, or $C_n^{-1/2} B_m C_n^{-1/2}$, for $0 \leqslant m \leqslant n-1$. By writing

$$C_n^{-1/2} A_m C_n^{-1/2} = C_n^{-1/2} C_m^{1/2} C_m^{-1/2} A_m C_m^{-1/2} C_m^{1/2} C_n^{-1/2}$$

$$C_n^{-1/2} B_m C_n^{-1/2} = C_n^{-1/2} C_m^{1/2} C_m^{-1/2} B_m C_m^{-1/2} C_m^{1/2} C_n^{-1/2}$$

and taking into account that $C_m^{-1/2} A_m C_m^{-1/2}$ and $C_m^{-1/2} B_m C_m^{-1/2}$ are converging sequences and that $C_n^{-1/2} C_m^{1/2} \leqslant I$ if $m \leqslant n$ ($(C_n)_n$ is an increasing matrix sequence) we conclude that the entries of $\tilde{J}_{nN}^{(n)}$ are bounded (independently of n). ∎

We complete this section by establishing a relationship between the zeros of the polynomial $P_n(t)$ and the zeros of the scaled polynomial $P_n(C_n; t)$.

Lemma 3.2. *For $j = 1, ..., nN$, there exist positive constants $\beta_{n,j}$ such that*

$$\mu_{n,1} \leqslant \beta_{n,j} \leqslant \mu_{n,N}, \qquad \text{and} \qquad x_{n,n,j} = \frac{x_{n,j}}{\beta_{n,j}},$$

where $x_{n,j}$ are the zeros of $P_n(t)$, $x_{n,n,j}$ the zeros of $P_n(C_n; t)$, and $\mu_{n,1} \leqslant \cdots \leqslant \mu_{n,N}$ the eigenvalues of C_n.

Proof. Taking into account that the zeros of $P_n(t)$ are the eigenvalues of J_{nN} (truncated N-Jacobi matrix of dimension nN), the zeros of $P_n(C_n; t)$ are the eigenvalues of the matrix $\tilde{J}_{nN}^{(n)}$ (see Lemma 3.1), and that

$$\tilde{J}_{nN}^{(n)} = \mathbf{C}_{\mathbf{n}}^{-1/2} J_{nN} \mathbf{C}_{\mathbf{n}}^{-1/2},$$

where $\mathbf{C}_{\mathbf{n}}$ is the $nN \times nN$ diagonal block matrix defined as

$$\mathbf{C}_{\mathbf{n}} = \begin{pmatrix} C_n & \theta & \cdots & \theta \\ \theta & C_n & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & C_n \end{pmatrix},$$

the Lemma follows by applying a well-known result by Ostrowski (see [HJ1, p. 224]) about the eigenvalues of SAS^*, where A is Hermitian and S is nonsingular. ∎

4. THE DEGENERATE CASE

We study here the case when the limit matrix A is singular. The ratio asymptotics also exists but this behaviour strongly depends on the structure of the matrix A. The situation is very similar to the degenerate case of convergence recurrence coefficients studied in [D6, Sect. 4].

We prove that if in the hypothesis of Theorem 1.1 we assume the limit matrix A to be singular and the sequence $(C_n)_n$ to be increasing (which implies that Γ is bounded), then there still exists a positive definite matrix of measures ν, which is degenerate (more precisely $\int (tI-B)\, d\nu(t)(tI-B)^*$ is singular), for which

$$\lim_{n\to\infty} C_n^{1/2} P_{n-1}(C_n; z)\, P_n^{-1}(C_n; z)\, A_n^{-1} C_n^{1/2} = \int \frac{d\nu(t)}{z-t}, \qquad z \in \mathbb{C}\setminus\Gamma. \tag{4.1}$$

Proceeding as in Section 3, we can reduce the result to the case of varying recurrence coefficients. Hence, it will be enough to prove that if in the hypothesis of Theorem 2.1 we assume the limit matrix A to be singular and Γ to be bounded, then there still exists a positive definite matrix of measures ν, which is degenerate (more precisely $\int (tI-B)\, d\nu(t)(tI-B)^*$ is singular), for which

$$\lim_{m\to\infty} R_{n_m-1,k_m}(z)\, R_{n_m,k_m}^{-1}(z)\, A_{n_m,k_m}^{-1} = \int \frac{d\nu(t)}{z-t}, \qquad \text{for} \quad z \in \mathbb{C}\setminus\Gamma. \tag{4.2}$$

Using the matrix polynomials $t^l I$ instead of $U_l^{A,B}(t)$ we find that the coefficients $\Delta_{i,l,n-1,k}$ which appear in (2.5) now satisfy the formula

$$\Delta_{j,l+1,n-1,k} = \Delta_{j,l,n-1,k} B_{n-j,k} + \Delta_{j-1,l,n-1,k} A^*_{n-j+1,k} \\ + \Delta_{j+1,l,n-1,k} A_{n-j,k},$$

instead of (2.6). Using induction on l, it is easy to prove that the limit $\lim_{m\to\infty} \Delta_{j,l,n_m-1,k_m}$ exists for $j, l \geqslant 0$, although in this case we cannot compute it explicitly.

This shows that the limits $\lim_{m\to\infty} \int t^l\, d\mu_{n_m,k_m}$, $l \geqslant 0$, exist. Since $\int d\mu_{n,k} = I$, $n, k \geqslant 0$ (see (2.4)) by using the Banach–Alaoglu theorem we conclude that $(\mu_{n_m,k_m})_m$ has a limit point ν. Since Γ is bounded this matrix of measures ν has compact support. We can now finish by applying again the *method of moments*.

To prove that this matrix of measures v is degenerate, it is enough to use the quadrature formula for the polynomials $(R_{n,k})_k$ and the definition of the measure v to get that (see [D6, Sect. 4] for more details)

$$\int (tI-B)\,dv(t)(tI-B)^* = A^*A.$$

Although we have not found an explicit expression for v, this matrix of measures can be computed from the following expression of its Hilbert transform. If we write

$$F_{A,B}(z) = \int \frac{dv(t)}{z-t}, \qquad z \notin \operatorname{supp}(v),$$

then this analytic matrix function satisfies the matrix equation

$$A^* F_{A,B}(z)\, A F_{A,B}(z) + (B - zI)\, F_{A,B}(z) + I = \theta.$$

Indeed, let us multiply to the right the formula given in (3.1) by $P_n^{-1}(C_n; z)\, C_n^{-1/2}$. We now put $k=n$ and take limit as n tends to ∞. First of all, let us notice that the ratio

$$C_n^{-1/2} A_{n+1} P_{n+1}(C_n; z)\, P_n^{-1}(C_n; z)\, C_n^{-1/2}$$

tends to $F_{A,B}^{-1}(z)$. To see this, it is enough to write it as

$$[D_{n+1}^{1/2} P_n(D_{n+1}; z)\, P_{n+1}^{-1}(D_{n+1}; z)\, A_{n+1}^{-1} D_{n+1}^{1/2}]^{-1},$$

where $D_{n+1} = C_n$, and take into account that the sequence $(D_n)_n$ satisfies the same hypothesis that the sequence $(C_n)_n$. Hence after taking the limit, we have

$$zI = F_{A,B}^{-1}(z) + B + A^* F_{A,B}(z)\, A,$$

from where it is easy to obtain the matrix equation.

5. EXAMPLES

(1) Let us consider the 2×2 case when the matrix limits A and B are diagonal matrices,

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix},$$

with $a, b > 0$. Taking into account Corollary 2.3 of [D6], we can give the following explicit expression for the ratio asymptotic of $(P_n)_n$:

$$\lim_n C_n^{1/2} P_{n-1}(C_n; z)\, P_n^{-1}(C_n; z)\, A_n^{-1} C_n^{1/2}$$

$$= \frac{1}{2} \begin{pmatrix} (z-c)/a^2 & 0 \\ 0 & (z-d)/b^2 \end{pmatrix}$$

$$- \frac{1}{2} \begin{pmatrix} \sqrt{(z-c)^2 - 4a^2}/a^2 & 0 \\ 0 & \sqrt{(z-d)^2 - 4b^2}/b^2 \end{pmatrix}.$$

(2) An important particular case of Example 1 is when the matrix recurrence coefficients have the form

$$B_0 = \begin{pmatrix} b_{-1} & a_0 \\ a_0 & b_0 \end{pmatrix},$$

$$B_n = \begin{pmatrix} b_{-n-1} & 0 \\ 0 & b_n \end{pmatrix}, \qquad n = 1, \ldots$$

$$A_n = \begin{pmatrix} a_{-n} & 0 \\ 0 & a_n \end{pmatrix}, \qquad n = 0, 1, \ldots.$$

They are closely related to bilateral birth and death processes (see [P, ILMV, V4]) and to the doubly infinite difference equation (see [MR]),

$$t\alpha_n(t) = a_{n+1}\alpha_{n+1}(t) + b_n\alpha_n(t) + a_n\alpha_{n-1}(t), \qquad n \in \mathbb{Z}.$$

The cases $a_n = dn$, $b_n^2 = an^2 + bn + c$, $n \in \mathbb{Z}$, with a, b, c, d real, $a, c \neq 0$ and $b_n^2 > 0$, were studied in [MR], and are related to associated Meixner $(d^2 > 4a > 0)$, Meixner–Pollaczek $(d^2 < 4a)$ and Laguerre $(d^2 = 4a \neq 0)$ polynomials.

For these particular examples, we can now take

$$C_n = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{n} \end{pmatrix}.$$

Since C_n is diagonal, we can define the scaled polynomials $P_n(C_n; t)$ using (1.3), (1.4), or (1.5).

The sequence $(C_n)_n$ satisfies

$$\lim_n C_n^{-1/2} C_{n-1}^{1/2} = I;$$

the other limits in (1.2) can also be easily computed:

$$\lim_n C_n^{-1/2} A_n C_n^{-1/2} = A = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix},$$

$$\lim_n C_n^{-1/2} B_n C_n^{-1/2} = B = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a} \end{pmatrix}.$$

From Example 1, we get the asymptotic behaviour

$$\lim_n \frac{1}{d} P_{n-1}(C_n; z)\, P_n^{-1}(C_n; z) = \frac{1}{2d^2} (z - \sqrt{a} - \sqrt{(\sqrt{a} - z)^2 - 4d^2})\, I.$$

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